ON THE (DE)HOMOGENIZATION OF SAGBI BASES FOR FREE ASSOCIATIVE ALGEBRAS

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Abstract. This paper has been inspired by a note on (De) homogenization of Sagbi (Subalgebra Analog to Gröbner Bases for Ideals) Bases in a polynomial ring over a field K [4]. It intends to study the behavior of Sagbi Bases for Free Associative Algebras under non-central (De)homogenization technique with respect to an additional variable. This paper demonstrates how Sagbi Bases behave differently in Free Associative Algebras under the said technique compared to Sagbi Bases in polynomial rings. The explanation has been made by illustrating a few examples as well.

1. Introduction

In Computational Commutative Algebra, polynomial subalgebra is the second most important topic in Ring theory after Ideals. Concept of Sagbi (Subalgebra Analog to Gröbner Bases for Ideals) Bases for subalgebra parallel the role of Gröbner Bases for Ideals. The theory of Sagbi bases was introduced by Robbiano and Sweedler [6] and independently by Kapur and Madlener. The motivation for this paper is also bought from [1] non- central (De)homogenization of Gröbner bases. Since many of the basic concepts of Gröbner bases also apply to subalgebra, extracting results of applying homogenization and dehomogenization on Sagbi Bases is natural. The commutative case has already been discussed in [4]; This paper addresses the non-central (De)homogenization of Sagbi bases in the case of non-commutative rings.

The paper herein comprises three sections. Section 2 introduces related terms, definitions, and notations, and section 3 consist of a detailed discussion on how (De) homogenization works in the case of Free Associative Algebra by computing Sagbi bases for (De)homogenized Subalgebras. In this section, we will present two important results (Theorem 3.5 and 3.6) about Sagbi Bases of homogenized and dehomogenized subalgebra.

2. Notation and definition

Let us review some terminologies and results of Sagbi bases theory for free associative algebras i.e $R = K\langle X\rangle = K\langle X_1, X_2, ...X_n\rangle$. The reader is expected to go through the whole section to make sure the notations used are clear.
In commutative Sagbi theory in which we have $R = K[X] = K[x_1, x_2, ...x_n]$, the role of monomials is played by exponent functions; i.e., for a monomial $X^\alpha = X_1^{\alpha_1}, X_2^{\alpha_2}, X_3^{\alpha_3}...X_n^{\alpha_n} \alpha_i \in \mathbb{N}$, we speak of exponent vector $\alpha = (\alpha_1, ...\alpha_n)$. Since

2010 *Mathematics Subject Classification.* Primary 13P10;
Key words and phrases. Sagbi basis, homogenous polynomial.

this approach is not possible in the non-commutative case, i.e., when $R = K\langle X\rangle$, we here use only monomials. The set of all monomials is represented as Mon(R). A Linear combination of these monomials with scalars from K forms a polynomial.
We have $K\langle X\rangle$, the free associative algebra generated by elements $X_1, X_2....X_n$. Each element $f \in K\langle X\rangle$ has a degree given by the cardinality of symbols composing it, so for, e.g., $deg(X_1X_2X_1) = 3$. For f written as a combination of more than one monomial, the highest degree of monomials in f will give the degree of f.
Let R_j be the K-vector space spanned by all homogeneous polynomials of degree j. The set of monomials u of vector space R_j with degree(u)=j form a K basis of this vector space and is of finite dimension.
If G is a subset of $K\langle X\rangle$(not necessarily finite), we use $K\langle G\rangle$ for the subalgebra of $K\langle X\rangle$ generated by G. This notation is natural since elements of $K\langle G\rangle$ are precisely the polynomials in the set of formal variables G viewed as elements of $K\langle G\rangle$.

Now we come to the ordering of elements. We choose the *deglex* ordering $\prec$ such that $u_1 \prec u_2$ iff either $deg(u_1) \prec deg(u_2)$ or $deg(u_1) = deg(u_2)$, and u_1 is lexicographically less than u_2, We say that u_1 is lexicographically less than u_2 if either there is $r \in K\langle X\rangle$ such that $u_2 = u_1 r$ or there are $l, r_1, r_2 \in K\langle X\rangle$, a_{j_1}, a_{j_2} with $j_1 \prec j_2$ such that $u_1 = la_{j_1}r_1, u_2 = la_{j_2}r_2$. Also, for each non-zero element $f \in K\langle X\rangle$, we associate $LW(f)$ as the leading word of f and $LC(f)$ as the leading coefficient of f. Whereas, for a subset $G \subset K\langle X\rangle$, we define $LW_\prec(G) = \{LW_\prec(f) | f \in G\}$.
(The above definition depends obviously on an ordering of the generating symbols).

We present an example here: let $K\langle X\rangle = \langle a_1, a_2\rangle$. Then we have:

$$1 \prec a_1 \prec a_2 \prec a_1^2 \prec a_1a_2 \prec a_2a_1 \prec a_2^2 \prec a_1^3 \prec a_1^2a_2 \prec a_1a_2a_1 \prec a_1a_2^2 \prec a_2a_1a_2...$$

It can be noticed that since it is not a well-ordering infact $a_1^{j+1}a_2^i \prec a_1^j a_2^{i+1}$, thus it can be remarked that "lexicographically less" is not good here.
We define extended lexicographic grading on $K\langle X, T\rangle$ as:

$$1 \prec_T T \prec_T a_1 \prec_T a_2 \prec_T a_1^2 \prec_T a_1a_2 \prec_T a_2a_1 \prec_T a_2^2 \prec_T a_1^3 \prec_T a_1^2a_2 \prec_T a_1a_2a_1 \prec_T a_1a_2^2 \prec_T a_2a_1a_2...$$

Now we gather some definitions of Sagbi bases for our reference. For a detailed study refer to [6].

Definition 2.1. A subset S of $K\langle G\rangle$ is called Sagbi basis of $K\langle G\rangle$ with respect to $\prec$ if

$$K\langle LM_\prec(K\langle G\rangle)\rangle = K\langle LM_\prec(S)\rangle$$

In other words for any $f \in K\langle G\rangle$ we have $m(S) \in Mon(S)$ such that $LM_\prec(f) = LM_\prec(m(S))$.

Though Algorithms were developed for finding the Sagbi basis for subalgebras, the Following result helped detect if a given subset of a subalgebra is a Sagbi basis.

Proposition 2.2. *([3], Proposition 4) Let $\mathcal{W} \subset K\langle X\rangle$ be such that $\mathcal{LW}_\prec(w_i) \neq \mathcal{LW}_\prec(w_j)$ for $w_i \neq w_j$, $w_i, w_j \in \mathcal{W}$. If no word in $\mathcal{LW}_\prec(\mathcal{W})$ is a prefix(proper left factor) of some other word in $\mathcal{LW}_\prec(\mathcal{W})$, or no word in $\mathcal{LW}_\prec(\mathcal{W})$ is a suffix(proper right factor) of some other, then $\mathcal{W}$ is a Sagbi basis.*

3. (De)homogenized Sagbi basis in Free Associative Algebras

For fixed $\mathbb{N}$-graded structures, $K\langle X\rangle = \oplus_{p\in\mathbb{N}} K\langle X\rangle_p$ and $K\langle X,T\rangle = \oplus_{p\in\mathbb{N}} K\langle X,T\rangle_p$. Consider a ring epimorphism,

$$\varphi : K\langle X,T\rangle \to K\langle X\rangle$$

defined as $\varphi(X_i) = X_i$ and $\varphi(T) = 1$. The surjectivity of the map implies that each f in $K\langle X\rangle$ is the image of some homogeneous element in $K\langle X,T\rangle$. In other words, if $f = f_p + f_{p-1} + ... + f_{p-s}$ with $f_j \in K\langle X\rangle_j$ and $f_p \neq 0$, then

$$\tilde{f} = f_p + Tf_{p-1} + ...T^s f_{p-s}$$

is a degree p homogeneous element in $K\langle X,T\rangle$ such that $\varphi(\tilde{f}) = f$. This $\tilde{f}$ obtained is called *non-central homogenization of* f with respect to T. Where T is a non-commuting variable. Further, if we select arbitrary element F from $K\langle X,T\rangle$, we apply dehomegenization of F as, $F_\sim = \varphi(F)$. This element $F_\sim$ is called as *non-central dehomogenization of* F *with respect to* T in $K\langle X\rangle$.

Lemma 3.1. *([1], lemma 2.2) Fixing the notations and assumptions described above, the following statements are valid:*
(i) For each non zero element s in $K\langle X\rangle$,

$$(s^\sim)_\sim = s,$$

(ii) If $s \in K\langle X\rangle$, then

$$\mathcal{LW}_\prec(s) = \mathcal{LW}_\prec(\mathcal{LH}(s)) \;\; w.r.t \;\prec$$

If $S \in K\langle X,T\rangle$, then

$$\mathcal{LW}_{\prec_T}(S) = \mathcal{LW}_{\prec_T}(\mathcal{LH}(S)) \;\; w.r.t \;\prec_T,$$

(iii) For each $s \neq 0$ in $K\langle X\rangle$, we have

$$\mathcal{LW}_\prec(s) = \mathcal{LW}_\prec(\tilde{s}) \;\; w.r.t \;\prec .$$

On defining these we now see how homogenization is applied to algebras. So, for a subalgebra $B = K\langle W\rangle$ i.e subalgebra generated by a subset W of $K\langle X\rangle$, we write

$$\tilde{W} = \{\tilde{w} \mid w \in W\} \cup \{X_iT - TX_i \mid 1 \leqslant i \leqslant n\},$$

and we let

$$W^* = \{\tilde{w} \mid w \in W\}$$

Note that $\tilde{W} = W^* \cup \mathcal{T}$ where $\mathcal{T} = \{X_iT - TX_i \mid 1 \leqslant i \leqslant n\}$.

Definition 3.2. Let B be a subalgebra in $K\langle X\rangle$ and D be a subalgebra in $K\langle X,T\rangle$.
(a) For $\tilde{B} = \{\tilde{f} \mid f \in B\} \cup \{X_iT - TX_i \mid 1 \leq i \leq n\}$, the subalgebra $\overline{B} = K\langle\tilde{B}\rangle$ in $K\langle X,T\rangle$ is called the homogenization of B with respect to the variable T.
(b) The set $D_\sim = \{f_\sim \mid f \in B\}$ is called the dehomogenization of subalgebra D. This $D_\sim$ is a subalgebra of $K\langle X\rangle$.

Now, we will see through an example that for a nonempty subset W of $K\langle X\rangle$ and $B = K\langle W\rangle$, in general, $K\langle\tilde{W}\rangle \subset \overline{B}$.

Example 3.3. *Let $B = K\langle W\rangle$ be a subalgebra generated by a subset W of $K\langle X, Y, Z\rangle$, where $W = \{f_1, f_2\}$ with $f_1 = XY^2 + XY$ and $f_2 = XY^2 + XZ$. Now it may be noted that $(f_1 - f_2)^{\sim} = (XY + XZ)$ is contained in $\overline{B}$, but not in the subalgebra $K\langle\tilde{W}\rangle = K\langle XY^2 + TXY, XY^2 + TXZ, XT - TX, YT - TY, ZT - TZ\rangle$ which does not contain any homogeneous polynomial of degree 2, not involving T.*

Before stating our main theorem, we discuss a few important theorems and lemma.

Theorem 3.4. *Let B be a subalgebra in $K\langle X\rangle$, then $(\overline{B})_{\sim} = B$.*

Proof 3.5. *We have $B \subset (\bar{B})_{\sim}$. Now let $f \in (\bar{B})_{\sim}$, by definition 3.2 of homogenized subalgebra, dehomogenization of $\bar{B}$ will gives a single set of the form $\{(\tilde{f})_{\sim} | f \in B\}$, which by 3.1(i) gives an element of B. And hence $f \in B$.*

Lemma 3.6. *The set $\mathcal{T} = \{X_iT - TX_i \mid 1 \leq i \leq n\}$ is a Sagbi Basis of $K\langle X, T\rangle$ with respect to any monomial ordering on $\prec_T$.*

Proof 3.7. *Since $\mathcal{LW}_{\prec_T}(X_iT - TX_i) = X_iT$, leading words(monomials) of elements of this set are neither prefixes nor suffixes of other words. Hence by the preposition 2.2, this set is a Sagbi Basis.*

The following theorem examines the behavior of Sagbi basis of subalgebras under non-central homogenization:

Theorem 3.8. *Let $B = K\langle S\rangle$ be the subalgebra of $K\langle X\rangle$ generated by a subset S and $\bar{B}$, the homogenization of the subalgebra B in $K\langle X, T\rangle$ with respect to T. The following two statements are equivalent*
(a) S is a Sagbi basis for B in $K\langle X\rangle$ with respect to monomial ordering $\prec$.
(b) $\tilde{S} = \{\tilde{s} \mid s \in S\} \bigcup \{X_iT - TX_i\}$ is a Sagbi basis for $\bar{B}$ in $K\langle X, T\rangle$ with respect to $\prec_T$.

Proof 3.9. *(a)$\Rightarrow$ (b) In order to prove that $\tilde{S}$ is a Sagbi basis for $\bar{B}$, We claim that for any $F \in \bar{B}$, there exists $m(\tilde{S})$ such that $\mathcal{LW}_{\prec_T}(F) = \mathcal{LW}_{\prec_T}(m(\tilde{S}))$.*
Since $\mathcal{LW}_{\prec_T}(F) = \mathcal{LW}_{\prec_T}(LH(F))$ lemma 3.1(iI) and $F \in \bar{B}$, we may assume that F is a non-zero homogeneous polynomial of $K\langle X, T\rangle$.
Now, the element F in $\overline{B}$ is a member of the subalgebra generated by $\tilde{B}$. It may be noticed that, in general, the leading monomial of F will be of form,

$$\mathcal{LW}_{\prec_T}(F) = u_1v_1u_2v_2...u_iv_j$$

where,

$$u_i = \mathcal{LW}_{\prec_T}(h_i) \quad for\ h_i \in \Omega\langle B^*\rangle$$
$$v_j = \mathcal{LW}_{\prec_T}(w_j) \quad for\ w_j \in \Omega\langle \mathcal{T}\rangle$$

It is sufficient to show that our claim is true when $\mathcal{LW}_{\prec_T}(F) = uv$ where $u = \mathcal{LW}_{\prec_T}(f)$ for some $f \in K\langle B^*\rangle$ and $v = \mathcal{LW}_{\prec_T}(w)$ for some $w \in K\langle\mathcal{T}\rangle$. By using lemma 3.1(iii) we have $u = \mathcal{LW}_{\prec_T}(f) = \mathcal{LW}_{\prec}(f_{\sim})$. Since S is a Sagbi basis for B, some $m_1(S)$ exists such that $\mathcal{LW}_{\prec}(f_{\sim}) = \mathcal{LW}_{\prec}(m_1(S))$. So we have $u = \mathcal{LW}_{\prec}(m_1(S))$. Again by the same lemma, we have $u = \mathcal{LW}_{\prec}(m_1(S)) = \mathcal{LW}_{\prec}(m_1(S^*))$[1] for some $m_1(S^*)$ where $S^* = \{\tilde{s} | s \in S\}$.
While on the other hand, since the set $\mathcal{T}$ is a Sagbi Basis by lemma 3.6, $\exists$ some $m_2(\mathcal{T})$ such that $v = \mathcal{LW}_{\prec_T}(w) = \mathcal{LW}_{\prec_T}(m_2(\mathcal{T}))$.

[1]The polynomial $m_1(S^*)$ is the non-central homogenization of the polynomial $m_1(S)$.

Now we let $m(\bar{S}) = (m_1(S^*)).(m_2(\mathcal{T}))$*, where* $\bar{S} = S^* \cup \mathcal{T}$*. Taking the leading monomial on both sides we have* $\mathcal{LW}_{\prec_T}(m(\bar{S})) = \mathcal{LW}_{\prec_T}[(m_1(S^*)).(m_2(\mathcal{T}))]$*, which gives*

$$\mathcal{LW}_{\prec_T}(m(\overline{S})) = \mathcal{LW}_{\prec_T}(m_1(S^*)).\mathcal{LW}_{\prec_T}((m_2(\mathcal{T})).$$

It can now be noticed that $\mathcal{LW}_{\prec_T}(m_1(S^*))$ *and* $\mathcal{LW}_{\prec_T}((m_2(\mathcal{T}))$ *are* u *and* v*, respectively. So, we have*

$$\mathcal{LW}_{\prec_T}(m(\bar{S})) = u.v = \mathcal{LW}_{\prec_T}(F).$$

(b)⇒ (a) Conversely we suppose $\tilde{S}$ *is a Sagbi basis for the homogenized algebra* $\bar{B}$ *of* B *in* $\Phi\langle T\rangle$*. Let* $f \in B$*, then* $\tilde{f} \in \bar{B}$*. Therefore,* $\mathcal{LW}_{\prec_T}(\tilde{f}) = \mathcal{LW}_{\prec_T}(m(\tilde{S}))$ *for some* $m(\tilde{S})$

Since $\mathcal{LW}_{\prec_T}(m(\tilde{S})) = \mathcal{LW}_{\prec}(m(\tilde{S})_{\sim}) = \mathcal{LW}_{\prec}(m(S))$ *and by lemma 3.1(ii)* $\mathcal{LW}_{\prec}(f) = \mathcal{LW}_{\prec_T}(\tilde{f})$*. Then it follows that*

$$\mathcal{LW}_{\prec}(f) = \mathcal{LW}_{\prec}(m(S)),$$

i.e., S *is a Sagbi basis for* B*.*

This theorem is an exciting tool for evaluating the Sagbi basis of homogenized subalgebra by just passing it onto the homogenized generators. This result thus worked in the same way as it did for non-central homogenization on a Gröbner basis. Let us see an example here

Example 3.10. *Let* $B = K\langle S\rangle$ *be a subalgebra generated by a subset* $S = \{s_1 = X^2Y + 1, s_2 = 2YX - Y, s_3 = X^3 - Y^2\}$ *of* $\Omega\langle X, Y\rangle$*. This set* S *by proposition 2.2 is a Sagbi Basis w.r.t monomial ordering* $X \prec Y$*. So, by theorem 3.8, Sagbi basis of homogenized subalgebra* $\bar{B}$ *is given as* $\{X^2Y + T^3, 2YX - TY, X^3 + TY^2\} \cup \{XT - TX, YT - TY\}$ *.*

But things did not work well for the case of dehomogenized subalgebras. We will see and explore through a simple example where given a homogeneous Sagbi basis of a homogeneous subalgebra its dehomgenization does not give a Sagbi basis of dehomogenized subalgebra.

Example 3.11. *Consider a set* $S = \{TYZ, TZY, T^2Y + T^2X, XT - TX, YT - TY, ZT - TZ\}$ *and* $D = K\langle S\rangle$ *a homogeneous subalgebra of* $K\langle X, Y, Z, T\rangle$*. The set* S *is a Sagbi basis with respect to ordering* $T \prec_{grlex} X \prec_{grlex} Y \prec_{grlex} Z$ *by the proposition 2.2. We will show that the set* $S_{\sim} = \{YZ, ZY, Y + X\}$ *is not a Sagbi basis of* $D_{\sim}$ *w.r.t ordering* $X \prec_{grlex} Y \prec_{grlex} Z$*.*

Let $s_1 = YZ, s_2 = ZY, s_3 = Y + X \in D_{\sim}$*. Furthermore, we take* $P(S_{\sim}) = s_1s_3 - s_3s_2 = (YZ)(Y + X) - (Y + X)(ZY) = YZY + YZX - YZY - XZY = YZX - XZY \in D_{\sim}$*. It may be noted that there does not exist any* $m(S_{\sim})$ *such that* $\mathcal{LW}_{\prec}(P(S_{\sim})) = YZX = \mathcal{LW}_{\prec}(m(S_{\sim}))$ *since the leading words of* s_i *do not contain* X*. Hence* $S_{\sim}$ *is not a Sagbi basis of* $D_{\sim}$*.*

Acknowledgment:
Thanks to Dr. Junaid Alam Khan and my institute (IBA) for motivating me to write my first academic writing in Mathematics.

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